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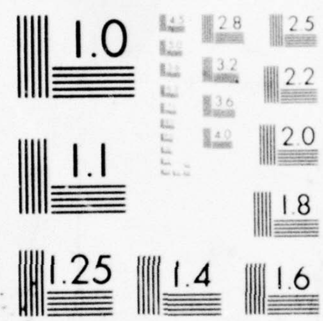


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# BIFURCATION FOR ODD POTENTIAL OPERATORS AND AN ALTERNATIVE TOPOLOGICAL INDEX

Edward R. Fadell and Paul H. Rabinowitz*

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## ABSTRACT

A bifurcation theorem is proved for odd potential operators. The operator equation (*) $f'(u) \equiv Lu + H(u) = \lambda u$ (LAMBDA) is treated where $\lambda \in \mathbb{R}$ (LAMBDA; is a member of) and $u \in E$ (is a member of), a real Hilbert space. A sharp description is given of the structure of the set of solutions of (*) near a bifurcation point as a function of $\lambda$ (LAMBDA). A crucial role is played here by a notion of topological index alternative to other indices used in critical point theory and the properties of this index are developed in some detail.

AMS (MOS) Subject Classifications: 47H15, 49F99, 55C99, 58E05

Key Words: bifurcation, odd potential operator, topological index, critical point, deformation theorem, minimax

Work Unit Number 1 (Applied Analysis)



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# BIFURCATION FOR ODD POTENTIAL OPERATORS AND AN ALTERNATIVE TOPOLOGICAL INDEX

Edward R. Fadell and Paul H. Rabinowitz*

## § 1. Introduction

In several recent papers [1-5], bifurcation theorems have been proved for potential operators. The purpose of this study is to prove a sharper result of this nature for odd potential operators. In doing so we will employ a topological index alternative to the notions of genus, Ljusternik-Schnirelman category, etc., which may also be of use in other problems.

To describe our work more fully, let $E$ be a real Hilbert space and $\Omega$ a neighborhood of $0$ in $E$. Suppose $f$ is a twice continuously Frechet differentiable real valued map on $\Omega$, i.e. $f \in C^2(\Omega, \mathbb{R})$ with $f(0) = 0$. Some standard remarks are in order. The Frechet derivative of $f$ at $u \in \Omega$, $f'(u)$, is a linear map from $E$ to $\mathbb{R}$ so $f'(u) \in E'$, the dual space of $E$. Since $E$ is self dual we can and will interpret the map $u \to f'(u)$ as a map from $E$ to $E$. We further assume $f'(u) = Lu + H(u)$ where $L$ is linear and $H(u) = o(\|u\|)$ at $u = 0$.

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For $\lambda \in \mathbb{R}$, consider the equation

$$f'(u) = \lambda u \; . \tag{1.1}$$

A solution of (1.1) is a pair $(\lambda, u) \in \mathbb{R} \times E$. Our above assumptions imply $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ are solutions of (1.1) and they shall be referred to as the <u>trivial solutions</u> of (1.1). A trivial solution $(\mu, 0)$ is called a <u>bifurcation point</u> if every neighborhood of $(\mu, 0)$ contains nontrivial solutions. It is well-known and easily shown that a necessary condition for $(\mu, 0)$ to be a bifurcation point is that $\mu \in \sigma(L)$, the spectrum of L. Under mild additional hypotheses, this necessary condition is also sufficient. (See e.g. [5] for references).

In some applications, e.g. to buckling problems in elasticity theory, solutions of (1.1) represent the possible equilibrium states of a physical system depending on a parameter $\lambda$. It is therefore of interest to study the solution set of (1.1) as a function of $\lambda$. Moreover in such problems it is often the case that $f$ is even and therefore solutions of (1.1) occur in pairs $(\lambda, \pm u)$. Our goal here is to give lower bounds for the number of nontrivial solutions of (1.1) near a bifurcation point as a function of $\lambda$ when $f$ is even. Our main result is:

<u>Theorem 1.2</u>: Let $E$ be a real Hilbert space, $\Omega$ a neighborhood of $0$ in $E$, and $f \in C^2(\Omega, \mathbb{R})$ where $f$ is even and $f'(u) = Lu + H(u)$ with $L$ linear and $H(u) = o(\|u\|)$ at $u = 0$. Suppose $\mu \in \sigma(L)$ is an isolated eigenvalue of $L$ of multiplicity $n < \infty$. Then either (i) $(\mu, 0)$

is not an isolated solution of (1.1) in $\{\mu\} \times E$ or (ii) there exist left and right neighborhoods, $\mathcal{J}_\ell$ and $\mathcal{J}_r$, of $\mu$ in $\mathbb{R}$ and integers $k, m \geq 0$ such that $k + m \geq n$ and if $\lambda \in \mathcal{J}_\ell$ (resp. $\mathcal{J}_r$), (1.1) possesses at least $k$ (resp. $m$) distinct pairs of nontrivial solutions. Moreover as $\lambda \to \mu$, these solutions converge to $(\mu, 0)$.

Remark 1.3: Either $\mathcal{J}_\ell$ or $\mathcal{J}_r$ may be empty. A characterization of $k$ and $m$ will be given in the course of the proof of the theorem.

Theorem 1.2 improves earlier results in this direction due to Clark [3] and Rabinowitz [5]. Other work on (1.1) for $f$ even has been carried out by Böhme [1] and Marino [2] who studied the solutions of (1.1) near $(\mu, 0)$ as a function of $\rho = \|u\|$. They showed in particular that under the hypotheses of Theorem 1.2, for each $\rho > 0$, there are at least $n$ distinct pairs of solutions $(\lambda(\rho), \pm u(\rho))$ of (1.1) having $\|u(\rho)\| = \rho$ and $(\lambda(\rho), u(\rho)) \to (\mu, 0)$ as $\rho \to 0$. Thus Theorem 1.2 is a natural complement to the Böhme-Marino result. We suspect that there is a better approach to (1.1) by means of which both Theorem 1.2 and the $\rho$ dependent result may be obtained simultaneously.

The proof of Theorem 1.2 will be given in § 2. In brief the main steps are: (1) Use a standard argument to reduce the problem of solving (1.1) near $(\mu, 0)$ to that of determining the critical points (with respect to $v$) of a function $g(\lambda, v)$ defined near $(\mu, 0)$ in $\mathbb{R} \times \mathbb{R}^n$; (2) Work in an appropriately defined neighborhood, $Q$, of $0$ in $\mathbb{R}^n$ to construct several families of sets $\Gamma_j$ in $\overline{Q}$ and study their properties;

(3) Minimax $g(\lambda, v)$ over each of these families of sets thereby producing a set of numbers; (4) Verify that each of these minimax values is a critical value of $g(\lambda, \cdot)$ and that we obtain the required number of critical points.

To define the sets in (2), a notion of topological index is introduced which plays a crucial role there and is of independent interest. To avoid unduly interrupting the proof of Theorem 1.2 in § 2, we state a lemma in § 2 which asserts the existence of an index with the properties we require and delay the definition of the index and development of its properties to § 3. The relationship of this index to others that have been employed earlier in critical point theory such as Ljusternik-Schnirelman category [6], coindex [7], genus [8] [9], and the indices of Yang [10-11] will also be discussed in § 3.

The authors acknowledge with thanks several helpful conversations with Charles Conley. In particular we are indepted to him for a suggestion which led to the final form of Theorem 3.14.

## § 2. The Main Theorem

In this section we will carry out the proof of Theorem 1.2. To begin, observe that although $E$ may be infinite dimensional, we can reduce (1.1) to a finite dimensional problem in a standard fashion using the method of Lyapunov-Schmidt. This has been done already e.g. in [5] but since it is brief we will include it here. Let $N \equiv N(L - \mu I)$, the null space of $L - \mu I$ and let $N^\perp$ denote its orthogonal complement in $E$. Since $N$ is $n$ dimensional, we can identify it with $\mathbb{R}^n$.

If $u \in E$, $u = v + w$ with $v \in N$ and $w \in N^\perp$. Letting $P$ and $P^\perp$ denote respectively the orthogonal projectors of $E$ onto $N$ and $N^\perp$, we see (1.1) is equivalent to the pair of equations:

$$(2.1)\qquad \begin{cases} \text{(i)}\ \ (\mu - \lambda)v + PH(v + w) = 0 \\ \text{(ii)}\ \ (L - \lambda I)w + P^\perp H(v + w) \equiv F(\lambda, v, w) = 0\ . \end{cases}$$

Note that $F(\mu, 0, 0) = 0$ and the Frechet derivative of $F$ with respect to $w$ at $(\mu, 0, 0)$, $F_w(\mu, 0, 0) = L - \mu I$ which is an isomorphism from $N^\perp$ to $N^\perp$. *Consequently by the implicit function theorem,* (2.1) (ii) can be solved for $w = \varphi(\lambda, v)$ in a neighborhood, $\mathcal{O}$, of $(\mu, 0) \in \mathbb{R} \times N$ with $\varphi \in C^1(\mathcal{O}, N^\perp)$. Since $f$ is even in $u$, it follows that $\varphi(\lambda, v)$ is odd in $v$. Moreover since $H(u) = o(\|u\|)$ at $u = 0$, (2.1) (ii) shows $\varphi(\lambda, v) = -(L - \lambda I)^{-1} P^\perp H(v + \varphi(\lambda, v)) = o(\|v\|)$ at $v = 0$ uniformly for $\lambda$ near $\mu$ (where the inverse is relative to $N^\perp$). Thus solving (1.1) for $(\lambda, u)$ near $(\mu, 0)$ in $\mathbb{R} \times E$ is equivalent to solving (2.1) (i) for $(\lambda, v)$ near $(\mu, 0)$ in $\mathbb{R} \times N$.

The next step in the proof is to define

$$g(\lambda, v) = f(v + \varphi(\lambda, v)) - \frac{\lambda}{2}(\|v\|^2 + \|\varphi(\lambda, v)\|^2) . \tag{2.2}$$

Note that $g$ is even in $v$ since $f$ is even and $\varphi$ is odd in $v$. A simple computation shows that for fixed $\lambda$, critical points of $g$ are solutions of (2.1) (i). Thus to prove Theorem 1.2, it suffices to determine lower bounds for the number of critical points of $g(\lambda, \cdot)$ near $v = 0$ for $\lambda$ fixed near $\mu$.

From (2.2),

$$g_v(\lambda, v) = (\mu - \lambda)v + PH(v + \varphi(\lambda, v)) . \tag{2.3}$$

The right hand side of (2.3) is continuously differentiable. Hence $g(\lambda, v)$ is a $C^2$ function of $v$ near $v = 0$ even though $\varphi(\lambda, v)$ and $f(v + \varphi(\lambda, v))$ are only continuously differentiable in $v$. Consider the ordinary differential equation:

$$\begin{cases} \dfrac{d\psi}{dt} = -g_v(\mu, \psi) \\ \\ \psi(0, x) = x \end{cases} \tag{2.4}$$

for $x$ near $0$ in $N$. If $v = 0$ is not an isolated critical point of $g(\mu, v)$, then we obtain (i) of Theorem 1.2. Thus now and henceforth we can assume there is a neighborhood, $V$, of $0$ in $N$ such that $0$ is the unique critical point of $g(\mu, v)$ in $V$.

<u>Lemma 2.5</u>: There is a constant $c > 0$ and a symmetric open neighborhood $Q$ of $0$, $Q \subset V$ such that $\overline{Q}$ is compact and

$1^{\circ}$ If $x \in Q$, $|g(\mu, x)| < c$.

$2^{\circ}$ If $x \in Q$, then $\psi(t, x) \in Q$ for all $t$ satisfying $|g(\mu, \psi(t, x))| < c$.

$3^{\circ}$ If $x \in \partial Q$, $|g(\mu, x)| = c$ or $\psi(t, x) \in \partial Q$ for all $t$ such that $|g(\mu, \psi(t, x))| \le c$.

Proof: The proof of Lemma 2.5 can be found in [5]. Q is simply the union of all orbit segments $\psi(t, x)$, for $x$ appropriately chosen near 0, which lie in $g(\mu, \cdot)^{-1}(-c, c)$ for $c$ sufficiently small.

Remark 2.6: For future reference observe that if $x \in \overline{Q}$, the orbit $\psi(t, x)$ can only leave $\overline{Q}$ by crossing $g(\mu, \cdot)^{-1}(-c)$. Note also that $\{x \in \overline{Q} \mid g(\mu, x) = c\}$ may be empty. This occurs when $g(\mu, \cdot)$ has an isolated local maximum at $v = 0$. Similarly $\{x \in \overline{Q} \mid g(\mu, x) = -c\}$ may be empty.

Given the existence of Q, we obtain a standard sort of "deformation theorem". For $z \in \mathbb{R}$, let $A_{\lambda z} = \{x \in \overline{Q} \mid g(\lambda, x) \le z\}$ and $K_{\lambda z} = \{x \in A_{\lambda z} \mid g(\lambda, x) = z,\ g_v(\lambda, x) = 0\}$.

Lemma 2.7: If $z \in \mathbb{R}$, $\varepsilon_1 > 0$, and U is any neighborhood of $K_{\lambda z}$, then there exists an $\varepsilon \in (0, \varepsilon_1)$ and an $\eta \in C([0,1] \times \overline{Q}, \overline{Q})$ such that:

$1^{\circ}$ $\eta(t, v)$ is odd in $v$.

$2^{\circ}$ $\eta(t, v) = v$ if $v \notin g(\lambda, \cdot)^{-1}[z - \varepsilon_1, z + \varepsilon_1]$.

$3^{\circ}$ $\eta(t, v)$ is a homeomorphism of $\overline{Q}$ to $\eta(t, \overline{Q})$ for each $t \in [0,1]$.

$4^{\circ}$ $\eta(1, A_{\lambda, z+\varepsilon} \setminus U) \subset A_{\lambda, z-\varepsilon}$.

$5^{\circ}$ If $K_{\lambda z} = \phi$, $\eta(1, A_{\lambda, z+\varepsilon}) \subset A_{\lambda, z-\varepsilon}$.

Proof: This lemma is the same as Lemma 1.19 in [5]. It is in the proof of this lemma that the special features of $Q$ play a role.

Next we require a suitable notion of index. We identify $N$ with $\mathbb{R}^n$ and set $B_\rho(y) = \{x \in \mathbb{R}^n \mid |x - y| < \rho\}$. Let $\mathcal{E}$ denote the set of compact subsets of $\mathbb{R}^n \setminus \{0\}$ which are symmetric with respect to the origin. $\mathbb{N}$ will denote the non-negative integers.

Lemma 2.8: There exists an index theory, i.e. a mapping $\mathcal{E} \to \mathbb{N}$, $A \to \text{Index } A$, possessing the following properties:

$1^\circ$ If $A = \phi$, Index $A = 0$; if $A \neq \phi$, Index $A \geq 1$; if $A = \{x, -x\}$, Index $A = 1$.

$2^\circ$ If $A, B \in \mathcal{E}$ and there is an odd map $\psi \in C(A, B)$, then Index $A \leq$ Index $B$. If $\psi$ is also a homeomorphism of $A$ onto $B$, then Index $A$ = Index $B$.

$3^\circ$ Index $(A \cup B) \leq$ Index $A$ + Index $B$.

$4^\circ$ If $A \in \mathcal{E}$, there exists a $\delta > 0$ and a uniform neighborhood of $A$, $N_\delta(A) = \{x \in \mathbb{R}^n \mid |x - A| \leq \delta\}$ such that Index $N_\delta(A)$ = Index $A$.

$5^\circ$ If $U$ is a symmetric bounded open neighborhood of $0$ in $\mathbb{R}^n$, Index $\partial U = n$.

$6^\circ$ Let $\rho > 0$, $K \in \mathcal{E}$ with $K \cap \overline{B_\rho(0)} = \phi$. Let $\tau > 0$ and suppose $\theta : K \times [0, \tau] \to \mathbb{R}^n \setminus \{0\}$ is an imbedding (i.e., $\theta$ is a one-one mapping) such that $\theta(x, 0) = x$, $x \in K$ and $\theta(\cdot, t)$ is odd on $K$ for each $t$. Then, if $\theta(K \times \{\tau\}) \subset B_\rho(0)$,

$$\text{Index}(\theta(K \times [0, \tau]) \cap \partial B_\rho(0)) = \text{Index } K .$$

We remark that it is the need for an index theory satisfying $6^\circ$ that requires us to go beyond the usual indices used in critical point theory, in particular, genus or Ljusternik-Schirelman category. We leave the precise definition of Index and the verification of its basic properties until § 3 and proceed now to complete the proof of Theorem 1.2 making use of Lemma 2.8.

Let $S^+ = \{x \in V \setminus \{0\} \mid \psi(x, t) \subset V \text{ for all } t > 0\}$ and $S^- = \{x \in V \setminus \{0\} \mid \psi(x, t) \subset V \text{ for all } t < 0\}$. It is not difficult to see that either $S^+$ or $S^-$ is nonempty [5]. In fact both are nonempty unless $v = 0$ is an isolated local maximum or minimum for $g(\mu, \cdot)$. Let $T^+ = S^+ \cap \partial Q$ and $T^- = S^- \cap \partial Q$. The proof of Theorem 1.2 is now a consequence of the following three results.

<u>Theorem 2.9</u>: Suppose Index $T^- = k > 0$. Then there is a left neighborhood $\mathcal{J}_\ell$ of $\mu$ such that for each $\lambda \in \mathcal{J}_\ell$, $g(\lambda, \cdot)$ possesses at least $k$ distinct pairs of nontrivial critical points. These points converge to $0$ as $\lambda \to \mu^-$.

<u>Corollary 2.10</u>: Suppose Index $T^+ = m > 0$. Then there is a right neighborhodd $\mathcal{J}_r$ of $\mu$ such that for each $\lambda \in \mathcal{J}_r$, $g(\lambda, \cdot)$ possesses at least $m$ distinct pairs of nontrivial critical points. These points converge to $0$ as $\lambda \to \mu^+$.

<u>Lemma 2.11</u>: Index $T^-$ + Index $T^+ \geq n$.

To establish Theorem 2.9, we require several families of sets, $\Gamma_j$, which are constructed next. Suppose Index $T^- = k$. For $K \subset T^-$ we define $\Phi(K) = \{\psi(t,x) \mid (t,x) \in (-\infty, 0) \times K\}$, i.e. we cone $K$ over 0 using the flow $\psi$. Now let $\mathfrak{F} = \{\chi \in C(\bar{Q}, \bar{Q}) \mid \chi$ is odd, one to one, and $\chi(v) = v$ if $v \in T^-\}$. For $1 \leq j \leq k$, define

$$G_j = \{\chi(\Phi(K)) \mid \chi \in \mathfrak{F},\ K \subset T^-, \text{ and Index } K \geq j\}.$$

Observe that $\theta \in \mathfrak{F}$ and $A \in G_j$ implies that $\theta(A) \in G_j$. Finally for $1 \leq j \leq k$, define

$$\Gamma_j = \{\overline{A \setminus Y} \mid A \in G_q \text{ for some } q,\ j \leq q \leq k,\ Y \in \mathcal{E}, \text{ and Index } Y \leq q - j\}.$$

<u>Lemma 2.12</u>: The sets $\Gamma_j$ possess the following properties:

$1^\circ$ $\Gamma_{j+1} \subset \Gamma_j$, $1 \leq j \leq k-1$.

$2^\circ$ If $\chi \in \mathfrak{F}$ and $B \in \Gamma_j$, then $\chi(B) \in \Gamma_j$.

$3^\circ$ If $B \in \Gamma_j$ and $Z \in \mathcal{E}$ with Index $Z \leq s < j$, then $B \setminus Z \in \Gamma_{j-s}$.

<u>Proof</u>: $1^\circ$ is obvious. To verify $2^\circ$, let $B \in \Gamma_j$. Therefore $B = \overline{A \setminus Y}$ with $A \in G_q$, $Y \in \mathcal{E}$, and Index $Y \leq q - j$. If $\chi \in \mathfrak{F}$, then $\chi(\overline{A \setminus Y}) = \overline{\chi(A \setminus Y)} = \overline{\chi(A) \setminus \chi(Y)}$. But $\chi(A) \in G_q$ by an above remark, $\chi(Y) \in \mathcal{E}$, and Index $\chi(Y) =$ Index $Y$ by $2^\circ$ of Lemma 2.8. Hence $\chi(B) \in \Gamma_j$. Finally to prove $3^\circ$, let $B = \overline{A \setminus Y}$ as in $2^\circ$. Therefore $\overline{B \setminus Z} = \overline{\overline{A \setminus Y} \setminus Z} = \overline{A \setminus (Y \cup Z)}$. Since $A \in G_q$ and Index $(Y \cup Z) \leq q - j + s = q - (j - s)$ by $3^\circ$ of Lemma 2.8, it follows that $\overline{B \setminus Z} \in \Gamma_{j-s}$.

<u>Proof of Theorem 2.9</u>: Define

$$c_j = \inf_{A \in \Gamma_j} \max_{v \in A} g(\lambda, v), \quad 1 \leq j \leq k. \tag{2.13}$$

By $1^o$ of Lemma 2.12, $c_1 \leq c_2 \leq \cdots \leq c_k$. We will further show: (i) $c_1 > 0$; (ii) $c_j$ is a critical value of $g(\lambda, \cdot)$ with a corresponding critical point in Q. (Since $c_1 > 0$, this critical point is nontrivial). (iii) If $c_{j+1} = \cdots = c_{j+p} \equiv d$, (i.e. d is what we might call a degenerate critical value of $g(\lambda, \cdot)$), then Index $K_{\lambda d} \geq p$. (iv) As $\lambda \to \mu^-$, any critical points corresponding to $c_j$, $1 \leq j \leq k$, converge to $v = 0$. By $1^o$ and $2^o$ of Lemma 2.8, if Index $A > 1$, A contains infinitely many distinct pairs of points. Hence Theorem 2.9 is a consequence of (ii) - (iv).

To prove (i), observe first from (2.2) that

$$(2.14) \qquad g(\lambda, v) = \frac{\mu - \lambda}{2} \|v\|^2 + \frac{1}{2}((L - \lambda I)\varphi(\lambda, v), \varphi(\lambda, v)) + h(v + \varphi(\lambda, v))$$

where $(\cdot, \cdot)$ denotes the inner produce in E, $h' = H$, and $h(0) = 0$. Since $\varphi(\lambda, v) = o(\|v\|)$ at $v = 0$ uniformly for $\lambda$ near $\mu$ and $h(u) = o(\|u\|^2)$ at $u = 0$, the dominating term in g for v near 0 is $\frac{\mu - \lambda}{2} \|v\|^2$. Therefore there is a $\rho > 0$, $\rho$ depending on $\lambda$, such that for $\lambda < \mu$ and $0 < \|v\| \leq \rho$,

$$(2.15) \qquad g(\lambda, v) \geq \frac{\mu - \lambda}{4} \|v\|^2 .$$

We can further assume $B_\rho(0) \cap \partial Q = \phi$. Now choose any $B \in \Gamma_1$. Then $B = \overline{\chi(\Phi(K)) \setminus Y}$ where $K \subset T^-$, Index $K = q \geq 1$, $Y \in \mathcal{E}$, and Index $Y \leq q - 1$. For $\tau$, depending on $\chi$ and K, sufficiently large, $\chi(\psi(-\tau, K)) \subset B_\rho(0)$. By $6^o$ of Lemma 2.8,

$$(2.16) \qquad \text{Index } \chi(\psi([-\tau, 0] \times K)) \cap \partial B_\rho(0) = \text{Index } K = q .$$

Now $2^o$ and $3^o$ of Lemma 2.8 together with (2.16) show

$$\text{(2.17)} \qquad \text{Index } B \cap \partial B_\rho(0) = \text{Index} \overline{[\chi(\Phi(K)) \cap \partial B_\rho(0)] \setminus Y} \geq$$

$$\geq \text{Index } \chi(\Phi(k)) \cap \partial B_\rho(0) - \text{Index } Y \geq q - (q-1) > 0 \; .$$

Therefore $1^o$ of Lemma 2.8 and (2.17) yield that $B \cap \partial B_\rho(0) \neq \phi$. Hence

$$\text{(2.18)} \qquad \max_{v \in B} g(\lambda, v) \geq \min_{\|v\| = \rho} g(\lambda, v) \geq \frac{\mu - \lambda}{4} \rho^2$$

via (2.15). Thus $c_1 \geq \frac{1}{4}(\mu - \lambda)\rho^2 > 0$ by (2.18).

To prove (ii), suppose that $c_j$ is not a critical value of $g(\lambda, \cdot)$. Then by Lemma 2.7 with $z = c_j$ and $\varepsilon_1 < z$, there is an $\varepsilon \in (0, \varepsilon_1)$ and a mapping $\theta(v) = \eta(1, v) \in C(\overline{Q}, \overline{Q})$ such that $\theta$ is odd in $v$ and

$$\text{(2.19)} \qquad \theta(A_{\lambda, c_j + \varepsilon}) \subset A_{\lambda, c_j - \varepsilon} \; .$$

For $\lambda$ near $\mu$, $g(\lambda, \cdot) < 0$ on $T^-$. Hence by $2^o$ and $3^o$ of Lemma 2.7, $\theta(v) = v$ for $v \in T^-$ and $\theta$ is 1 - 1 on $\overline{Q}$. Therefore $\theta \in \mathfrak{F}$. Choose $B \in \Gamma_j$ so that

$$\text{(2.20)} \qquad \max_{v \in B} g(\lambda, v) \leq c_j + \varepsilon \; .$$

By $2^o$ of Lemma 2.12, $\theta(B) \in \Gamma_j$. Consequently

$$\text{(2.21)} \qquad \max_{v \in \theta(B)} g(\lambda, v) \geq c_j \; .$$

But (2.21) contradicts (2.19) - (2.20) so $c_j$ is a critical value of $g(\lambda, \cdot)$.

A similar argument establishes (iii). Suppose Index $K_{\lambda d} < p$. By $4^o$ of Lemma 2.8, there is a $\delta > 0$ so that Index $N_\delta(K_{\lambda d})$ = Index $K_{\lambda d} < p$. Invoking Lemma 2.7 again with $z = d$ and $\varepsilon_1 < d$, there exists an $\varepsilon \in (0, \varepsilon_1)$ and an odd map $\theta(v) = \eta(1, v) \in C(\overline{Q}, \overline{Q})$ such that $\theta \in \mathfrak{F}$ and

$$(2.22) \qquad \theta(A_{\lambda, d+\varepsilon} \backslash N_\delta(K_{\lambda, d})) \subset A_{\lambda, d-\varepsilon} .$$

Choose $B \in \Gamma_{j+p}$ so that

$$(2.23) \qquad \max_{v \in B} g(\lambda, v) \leq d + \varepsilon = c_{j+p} + \varepsilon .$$

By $3^{\circ}$ of Lemma 2.12, $\overline{B \backslash N_\delta(K_{\lambda d})} \in \Gamma_{j+1}$ and by $2^{\circ}$ of the same lemma, $\theta(B \backslash N_\delta(K_{\lambda d})) \equiv M \in \Gamma_{j+1}$. Therefore

$$(2.24) \qquad \max_{v \in M} g(\lambda, v) \geq d = c_{j+1} .$$

But (2.24) contradicts (2.22) - (2.23).

Finally to prove (iv), observe that $g(\lambda, v) \to g(\mu, v)$ uniformly for $v \in \overline{Q}$ as $\lambda \to \mu$. Moreover $\overline{\Phi(T^-)} \in \Gamma_j$ for $1 \leq j \leq k$ and if $v \in \Phi(T^-)$, $g(\mu, v) < 0$. Since $0 \in \overline{\Phi(T^-)}$,

$$\max_{v \in \overline{\Phi(T^-)}} g(\mu, v) = 0 .$$

Therefore as $\lambda \to \mu^-$,

$$0 < c_j(\lambda) \leq \max_{v \in \overline{\Phi(T^-)}} g(\lambda, v) \to 0 .$$

Thus if $v_j(\lambda)$ is a critical point of $g(\lambda, \cdot)$ in $Q$ with $g(\lambda, v_j(\lambda)) = c_j(\lambda)$, we can find a sequence $\lambda_s \to \mu$ so that $v_j(\lambda_s) \to v$ with $g(\mu, v) = 0$ and $g_v(\mu, v) = 0$. But $0$ is the unique critical point of $g(\mu, \cdot)$ in $\overline{Q}$. Hence as $\lambda \to \mu$, $v_j(\lambda) \to 0$. The proof of Theorem 2.9 is now complete.

<u>Proof of Corollary 2.10</u>: Replace $g(\lambda, v)$ by $-g(\lambda, v)$. The result is then immediate from Theorem 2.9.

Proof of Lemma 2.11: The proof is based on that of Lemma 2.7 of [5]. Let $\rho > 0$ with $B_\rho(0) \subset V$. By Lemma 2.5 with $V$ replaced by $B_\rho(0)$, we can find a neighborhood $Q_b$ of $0$ having the same properties as $Q$ with $c$ replaced by $b$. If $v \in \partial Q_b \cap g(\mu,\cdot)^{-1}(-b)$, there is a unique $\tau(v) > 0$ so that $g(\mu, \psi(\tau(v), v)) = -c$. Moreover the map $\theta(v) = \psi(\tau(v), v)$ is odd and is in $C(\partial Q_b \cap g(\mu,\cdot)^{-1}(-b),\ \partial Q \cap g(\mu,\cdot)^{-1}(-c))$ with $\theta(S^- \cap \partial Q_b) = T^-$. Since Index $T^- = k$, by $2^\circ$ and $4^\circ$ of Lemma 2.8, there is a $\delta > 0$ so that Index $(N_\delta(T^-) \cap \partial Q) = k$. We claim for $\rho$ sufficiently small, $\theta(\partial Q_b \cap g(\mu,\cdot)^{-1}(-b)) \subset N_\delta(T^-) \cap \partial Q$. For otherwise there exist sequences $\rho_m \to 0$, $b_m \to 0$, and $x_m \in B_{\rho_m}(0)$ such that $g(\mu, x_m) = b_m > 0$ and $\psi(\tau(x_m), x_m) \in \partial Q$ but $x_m \notin N_\delta(T^-)$. Clearly along a subsequence $\psi(\tau(x_m), x_m) \to y \in \partial Q$ and $y \notin T^-$. But since $x_m \to 0$, $\tau(x_m) \to \infty$. Therefore $y \in T^-$, a contradiction, and we can find $\rho$ as above.

By $2^\circ$ of Lemma 2.8,

$$
\begin{aligned}
(2.25)\qquad \text{Index } S^- \cap \partial Q_b &= \text{Index } T^- = k \le \\
&\text{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(-b)) \le \text{Index } N_\delta(T^-) \cap \partial Q = k\,.
\end{aligned}
$$

Hence all inequalities in (2.25) are equalities. Similarly

$$
(2.26)\qquad \text{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(b)) = m = \text{Index } T^+\,.
$$

If $v \in \partial Q_b \backslash g(\mu,\cdot)^{-1}(-b)$, there is a unique $t(v) \le 0$ so that $g(\mu, \psi(t(v), v)) = b$. It follows that $\xi(v) = \psi(t(v), v)$ is a continuous odd map of $\partial Q_b \backslash g(\mu,\cdot)^{-1}(-b)$ onto $\partial Q_b \cap g(\mu,\cdot)^{-1}(b)$. Hence by $2^\circ$ of Lemma 2.8 again,

$$(2.27)\qquad \operatorname{Index}(\partial Q_b \setminus \overline{g(\mu,\cdot)^{-1}(-b)}) \le$$

$$\operatorname{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(b)) = m \le \operatorname{Index}(\partial Q_b \setminus g(\mu,\cdot)^{-1}(-b)) .$$

Thus we have equality in (2.27). Combining (2.25), (2.27) and $3^o$ and $5^o$ of Lemma 2.8 yields

$$(2.28)\qquad n = \operatorname{Index} \partial Q_b \le \operatorname{Index}(\partial Q_b \setminus \overline{g(\mu,\cdot)^{-1}(-b)}) + $$
$$+ \operatorname{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(-b)) = m + k$$

and the proof of Lemma 2.11 is complete.

## § 3. Definition and Properties of Index

The concepts of (Ljusternik-Schnirelmann) category as well as that of genus (called B-index by Yang [10] and coindex by Conner-Floyd [7]) have played a useful role in problems involving the existence of critical points. We develop here an alternative notion which is equivalent in a restricted category to the index introduced by Yang [11], and which has the properties usually enjoyed by these notions as well as one important additional one (Theorem 3.12 below). These properties were used in § 2 and summarized in Lemma 2.8, with $6^o$ corresponding to Theorem 3.12 below.

We work with the category $\mathcal{C}$ of compact metric spaces which admit a free $\mathbb{Z}_2$-action. More precisely, an object of $\mathcal{C}$ is a pair $(X, T)$ where $X$ is a compact metric space and $T : X \to X$ is a fixed point free homeomorphism of period 2. The morphisms of $\mathcal{C}$ are equivarient maps. i.e. given $(X, T)$ and $(X', T')$ in $\mathcal{C}$ a morphism $f : (X, T) \to (X', T')$ is a (continuous) map $f : X \to X'$ such that $f(Tx) = T'f(x)$, for $x \in X$. Thus, compact symmetric subsets of a normed linear space are then objects in $\mathcal{C}$ and odd maps between such subsets are morphisms in $\mathcal{C}$. A fortiori, then the category $\mathscr{C}$ of symmetric subsets of some $\mathbb{R}^n \setminus 0$ is included in $\mathcal{C}$.

Given $(X, T) \in \mathcal{C}$, $\tilde{X} = X/T$ is the corresponding orbit space and the map $q : X \to \tilde{X}$ which identifies $x$ and $Tx$ is a 2-fold covering map.

As usual, we will denote by $S^\infty$, the direct limit of the sequence of spheres of ascending dimension $S^1 \subset S^2 \subset S^3 \subset \cdots$, i.e., $S^\infty = \bigcup_{k=1}^{\infty} S^k$. $S^\infty$ admits the antipodal action and $P^\infty$, the corresponding infinite dimensional projective space, is on one hand the orbit space $S^\infty/T$, and on the other, the direct limit of the projective spaces $P^1 \subset P^2 \subset P^3 \subset \cdots$. It is easy to see that there exist equivariant maps $f : X \to S^\infty$ (in fact into $S^N$ for $N$ large) and any such map induces a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S^\infty \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{f}} & P^\infty \end{array}$$

where the vertical maps are the 2-fold covering maps and $\tilde{f}$ is naturally induced by $\tilde{f}$. We call any such $(f, \tilde{f})$ a <u>classifying</u> <u>map</u> for $(X, q, \tilde{X})$.

<u>Remark 3.1</u>: Both $S^\infty$ and $P^\infty$ receive the weak (= direct limit, = inductive) topology. For example, $U \subset S^\infty$ is open if, and only if $U \cap S^k$ is open in $S^k$ for all $k = 1, 2, \ldots$. It then follows easily that every compact subset of $S^\infty (P^\infty)$ lies in some $S^k (P^k)$ for $k$ sufficiently large.

<u>Remark 3.2</u>: We employ Čech cohomology with $\mathbb{Z}_2$ coefficients and the notation $H^q(X)$ stands for $H^q(X, \mathbb{Z}_2)$. We also use the fact that the $\mathbb{Z}_2$ cohomology of the real projective space $P^n$ is the polynomial ring over $\mathbb{Z}_2$ on one indeterminate $u \in H^1(P^n)$, truncated by the relation

$u^{n+1} = 0$. Recall also that the inclusion map $i : P^n \to P^{n+1}$ indices an isomorphism $i^* : H^q(P^{n+1}) \to H^q(P^n)$ for $q \leq n$.

We now give the definition of index which we will employ. Let $(X, T)$ denote an object of $\mathcal{C}$, as above, and let $(f, \tilde{f})$ denote a classifying map and $N$ chosen so that $f(X) \subset S^N$. Then set $\varphi(f, \tilde{f})$ equal to the max $k$ such that $\tilde{f}^*(u^k) \neq 0$ where

$$\tilde{f}^* : H^k(P^N) \to H^k(\tilde{X})$$

is induced by $\tilde{f} : \tilde{X} \to P^N$. Observe that $\varphi(f, \tilde{f})$ is independent of $N$ and that $\varphi(f, \tilde{f}) \leq \dim X$.

<u>Proposition-Definition 3.3</u>: Set

$$\text{index } X = \varphi(f, \tilde{f})$$

*for any classifying map* $(f, \tilde{f})$, [or alternatively for any equivariant map $f : X \to S^\infty$]. Then, index $X$ is independent of the choice of $(f, \tilde{f})$.

<u>Proof</u>: In order to prove independence of $(f, \tilde{f})$ let $(g, \tilde{g})$ denote another classifying map and choose $N$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & S^N \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{f}} & P^N \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & S^N \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{g}} & P^N \end{array} .$$

We imbed $X$ in the Hilbert cube $Q^\omega$. If $\eta : X \to Q^\omega$ is such an imbedding, then $\zeta : X \to Q^\omega \times Q^\omega$ defined by $\zeta(x) = (x, Tx)$ is an *equivariant imbedding using the action* $S(u, v) = (v, u)$ on $Q^\omega \times Q^\omega$. Recall now that $\eta(X)$ in $Q^\omega$ can be approximated by polyhedra in the following sense: for every $\varepsilon > 0$ there is a set $K_\varepsilon$ such

$\eta(X) \subset \operatorname{int} K_\varepsilon \subset K_\varepsilon \subset U_\varepsilon \subset Q^\omega$ where $U_\varepsilon$ is the $\varepsilon$-neighborhood of $X$, and $K_\varepsilon$ is homeomorphic to $P_\varepsilon \times Q^\omega$ where $P_\varepsilon$ is a *finite polyhedron*.

A simple modification of this yields the following

Lemma 3.4: For every $\varepsilon > 0$ there is an invariant set $K_\varepsilon \subset Q^\omega \times Q^\omega$, (i.e. $(u, v) \varepsilon K_2 \Longleftrightarrow (v, u) \varepsilon K_\varepsilon$) on which $S$ acts freely such that

$$\zeta(X) \subset \operatorname{int} K_\varepsilon \subset K_\varepsilon \subset U_\varepsilon \subset Q^\omega \times Q^\omega$$

where $U_\varepsilon$ is the $\varepsilon$-nghd of $\zeta(X)$ in $Q^\omega \times Q^\omega$ and $K_\varepsilon$ is homeomorphic to $P_\varepsilon \times Q^\omega$ where $P_\varepsilon$ is a finite polyhedron.

Now, using the above lemma we may identify $X$ with $\zeta(X)$ and $T$ with $S$, so that $X \subset Q^\omega \times Q^\omega$. We may extend the equivariant maps $f$ and $g$ to a neighborhood $V$ of $X$ in $Q^\omega \times Q^\omega$ and hence to equivariant maps

$$F : K_\varepsilon \to S^N \qquad G : K_\varepsilon \to S^N$$

where $X \subset K_\varepsilon \subset V$ and $K_\varepsilon$ is homeomorphic to $P_\varepsilon \times Q^\omega$, as in the above lemma. Now, we may appeal to the fact that $S^\infty \to P^\infty$ is a universal principal $\mathbb{Z}_2$-bundle to prove that $\tilde{F} \sim \tilde{G} : \tilde{K}_\varepsilon \to P^\infty$. Alternatively, working separately on the components of $K_\varepsilon$, one shows that

$$\tilde{F}_\# = \tilde{G}_\# : \pi_1(\tilde{K}_\varepsilon) \to \pi_1(P^\infty),$$

where $\tilde{F}_\#$ and $\tilde{G}_\#$ are the homomorphisms induced by $\tilde{F}$ and $\tilde{G}$, and then this forces $\tilde{F} \sim \tilde{G}$ since $P^\infty$ is a $K(\mathbb{Z}_2, 1)$ (see [12], pg. 427). *Hence, for a large positive integer* $M$ we have

$$\tilde{f} \sim \tilde{g} : \tilde{X} \to P^M$$

and hence $\tilde{f}^* = \tilde{g}^* : H^*(P^M) \to H^*(\tilde{X})$ and thus $\varphi(f, \tilde{f}) = \varphi(g, \tilde{g})$.

Remark 3.5: We adopt the convention that the index of the null set is $-1$ and if $X$ is a non-empty set in $C$ with $\tilde{f}^*(u) = 0$ above, then index $X = 0$. Also, notice that $\tilde{f}^*(u^k) = 0$ implies $\tilde{f}^*(u^\ell) = 0$ for $\ell > k$. We might also note here that a more inclusive notation would be index $(X, T)$ rather than index $X$, since $T$ plays a vital role. However, $T$ is not usually displayed, by convention.

We now investigate the basic properties of this index.

Proposition 3.6: index $X \leq \dim X$.

Proof: This is immediate because $H^q(X) = 0$ for $q > \dim X$, where $\dim X$ refers to the covering dimension of $X$ [13].

Proposition 3.7: If $g : X \to Y$ is equivariant, i.e. if $g$ is a morphism of the category $C$, then index $X \leq$ index $Y$.

Proof: Let $(f, \tilde{f})$ denote a classifying map for $Y$. Then, we have the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & S^\infty \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{Y} & \xrightarrow{\tilde{f}} & P^\infty \end{array}$$

where $(h = fg, \tilde{h} = \tilde{f}\tilde{g})$ is a classifying map for $X$. If index $Y = k$, then for $j > k$

$$\tilde{h}^*(u^j) = \tilde{g}^*\tilde{f}^*(u^j) = 0$$

and hence index $X \leq k =$ index $Y$.

Corollary 3.8: If $X \subset Y$, then $\text{index } X \leq \text{index } Y$.

Proposition 3.9: Let $K_1 \supset K_2 \supset \cdots \supset K_p \supset K_{p+1} \supset \cdots$ denote a descending sequence of compacta in $\Gamma$ with $X = \bigcap K_p$ and all receiving their free $\mathbb{Z}_2$-action by restricting that of $K_1$. Then, for some $p_0$, $\text{index } K_p = \text{index } X$, $p \geq p_0$.

Proof: We know that $\text{index } X \leq \text{index } K_p$ for every $p$, since $X \subset K_p$. Therefore, it suffices to show that for some $p_0$, $\text{index } K_p \leq \text{index } X$ for $p \geq p_0$. Given an equivariant map $f : X \to S^N \subset S^\infty$, we may extend $f$ to a neighborhood (in $K_1$) of $X$ and hence we may assume without loss that $f$ extends to $F : K_1 \to S^N \subset S^\infty$. Let $F_p = F|K_p$ and consider the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{i}_p"] \arrow[dr, "\tilde{i}_{p+1}"'] & \tilde{K}_p \arrow[r, "\tilde{F}_p"] & P^N \\
& \tilde{K}_{p+1} \arrow[u, "\tilde{j}_{p+1}"] \arrow[ur, "\tilde{F}_{p+1}"'] &
\end{tikzcd}$$

where $i_p : X \subset K_p$ and $j_{p+1} : K_{p+1} \subset K_p$ are inclusion maps. Then, we have an induced diagram

$$H^q(X) \xleftarrow{\alpha} \varinjlim H^q(K_p) \xleftarrow{\beta} H^q(P^N)$$

where $\alpha = \varinjlim \tilde{i}_p^*$ is an isomorphism using the continuity property of Čech theory, $p = \varinjlim \tilde{F}_p^*$ and $\alpha \circ \beta = f^* : H^q(P^N) \to H^q(\tilde{X})$. Suppose now that $\text{index } X = k$. Then, since $\tilde{f}^*(u^{k+1}) = 0$ and $\alpha$ is an isomorphism it follows that $\tilde{F}_{p_0}^*(u^{k+1}) = 0$ for some $p_0$ and hence for every $p \geq p_0$. Thus, index $K_p \leq k$ for all $p \geq p_0$ and the result follows.

Corollary 3.10: If $X \in \mathcal{C}$ is a subset of $\mathbb{R}^n \setminus \{0\}$, there is a symmetric polyhedron $K$ in $\mathbb{R}^n \setminus 0$ such that $X \subset$ interior $K$ and index $X$ = index $K$. $K$ may be chosen within any neighborhood of $X$ and in fact $K$ may be chosen as a smooth n-manifold with boundary.

Proof: Given a neighborhood $W$ of $X$ choose a sufficiently fine smooth triangulation of $\mathbb{R}^n \setminus \{0\}$ and let $K$ denote a regular neighborhood of an appropriate subpolyhedron containing $X$.

Corollary 3.11: If $(X, T) \in \mathcal{C}$, then $X$ may be equivariantly imbedded in $Q^\omega \times Q^\omega$ using the flip action $S(u, v) = (v, u)$ on $Q^\omega \times Q^\omega$. Identifying $X$ with its image in $Q^\omega \times Q^\omega$ and $T$ with $S$, there is a compact invariant set $K \subset Q^\omega \times Q^\omega$ such that $X \subset$ int $K$, index $X$ = index $K$ and $K$ is homeomorphic to $\underline{P} \times Q^\omega$ where $\underline{P}$ is a finite polyhedron. $K$ may be chosen within any neighborhood of $X$.

Proof: Apply Lemma 3.4.

Proposition 3.12: Suppose $X = A \cup B$, with $A$, $B$, and $X$ in $\mathcal{C}$ and where $A$ and $B$ receive their free $\mathbb{Z}_2$-actions from $X$. Then,

$$\text{index } X \leq \text{index } A + \text{index } B + 1 .$$

Proof: We will make use of the cup product in Čech theory over $\mathbb{Z}_2$ (see [14], p. 288)

$$H^p(X, A) \otimes_{\mathbb{Z}_2} H^q(X, B) \to H^{p+q}(X, A \cup B) .$$

Suppose index $A = p$, index $B = q$ and index $X = k$. Let $(f, \tilde{f})$ be a classifying map for X, with $(f_1, \tilde{f}_1)$ and $(f_2, \tilde{f}_2)$ serving as classifying maps for A and B, respectively, where $f_1 = f|A$ and $f_2 = f|B$. Then, for N sufficiently large, we have the diagram

$$\begin{tikzcd}
 & H^m(P^N) \arrow[dl, "\tilde{f}_2^*"'] \arrow[d, "\tilde{f}^*"] \arrow[dr, "\tilde{f}_1^*"] & \\
-H^m(\tilde{B}) & H^m(\tilde{X}) \arrow[l, "\tilde{i}^*"] \arrow[r, "\tilde{j}^*"'] & H^m(\tilde{A})
\end{tikzcd}$$

and exact sequences for pairs

$$\cdots \longrightarrow H^m(\tilde{X}, \tilde{A}) \xrightarrow{\alpha^*} H^m(\tilde{X}) \xrightarrow{\tilde{i}^*} H^m(\tilde{A}) \xrightarrow{\delta} H^{m+1}(\tilde{X}, \tilde{A}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^m(\tilde{X}, \tilde{B}) \xrightarrow{\beta^*} H^m(\tilde{X}) \xrightarrow{\tilde{j}^*} H^m(\tilde{B}) \xrightarrow{\delta} H^{m+1}(\tilde{X}, \tilde{B}) \longrightarrow \cdots .$$

Since

$$0 = \tilde{f}_1^*(u^{p+1}) = \tilde{i}^*\tilde{f}^*(u^{p+1})$$

$$0 = \tilde{f}_2^*(u^{q+1}) = \tilde{j}^*\tilde{f}^*(u^{q+1})$$

we have $x \in H^{p+1}(\tilde{X}, \tilde{A})$, $y \in H^{q+1}(\tilde{X}, \tilde{B})$ such that

$$\alpha^*(x) = \tilde{f}^*(u^{p+1}), \quad \beta^*(y) = \tilde{f}^*(u^{q+1}) .$$

Now, using the naturality of the cup product;

$$\begin{tikzcd}
H^{p+1}(\tilde{X}, \tilde{A}) \otimes H^{q+1}(\tilde{X}, \tilde{B}) \arrow[r] \arrow[d] & H^{p+q+2}(\tilde{X}, \tilde{A} \cup \tilde{B}) \arrow[d] \\
H^{p+1}(\tilde{X}) \otimes H^{q+1}(\tilde{X}) \arrow[r] & H^{p+q+2}(\tilde{X})
\end{tikzcd}$$

we see that $x \cup y = 0$ implies

$$0 = \tilde{f}^*(u^{p+1}) \cup \tilde{f}(u^{q+1}) = \tilde{f}^*(u^{p+q+2}) .$$

Therefore, $k \leq p + q + 1$ and the proof is complete.

<u>Proposition 3.13</u>: If $U$ is a bounded symmetric open set in $\mathbb{R}^{n+1}$ containing the origin with boundary $B = \partial U$, then

$$\text{index } B = n .$$

<u>Proof</u>: One considers, as usual, the odd map $f : B \to S^n$ which takes $x$ to $x/\|x\|$. This map induces an <u>injection</u>

$$\tilde{f}^* : H^q(P^n) \to H^q(\tilde{B}), \quad q \leq n .$$

The proof that $\tilde{f}^*$ is an injection is more or less classical and may be effected by using the transfer map (see Dold [14, p. 309]) as follows. First, we may assume that $f$ is extended to an odd map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f^{-1}(S^n) = B$. If we let $N^{n+1}$ denote $\mathbb{R}^{n+1}\setminus\{0\}$ with antipodes identified then $f$ induces $\tilde{f} : N^{n+1} \to N^{n+1}$ with $\tilde{f}^{-1}(P^n) = \tilde{B}$ and $\tilde{f}_*(o_{\tilde{B}}) = o_{P^n}$ where $o_{\tilde{B}} \in H_{n+1}(N^{n+1}, N^{n+1}\setminus\tilde{B})$, $o_{P^n} \in H_{n+1}(N^{n+1}, N^{n+1}\setminus P^n)$ are fundamental classes over $\mathbb{Z}_2$. Then, according to [14], there is a transfer map (over $\mathbb{Z}_2$)

$$\tilde{f}_! : H^q(\tilde{B}) \to H^q(P^n)$$

which acts as a right inverse for $\tilde{f}^* : H^q(P^n) \to H^q(\tilde{B})$. Thus, $\tilde{f}^*$ is an injection and this forces index $B \geq n$. Finally, since index $B \leq \dim B = n$, we have the desired result.

We now proceed to verify an important additional geometric property of index as defined above and which corresponds to $6^\circ$ of Lemma 2.8.

Theorem 3.14: Assume the following:

(i) $M^{n-1}$ is a compact connected symmetric manifold in $\mathbb{R}^n \setminus \{0\}$ separating $\mathbb{R}^n$ into components $U$ and $V$.

(ii) $A$ is a symmetric compact subset of $U$.

(iii) $\varphi : A \times [0, \tau] \to \mathbb{R}^n \setminus \{0\}$ is a symmetric imbedding ($\varphi(-x, t) = -\varphi(x, t)$) such that $\varphi(a, 0) = a$, $a \in A$ and $\varphi(A \times \tau) \subset V$.

Then, if we set $C = M^{n-1} \cap \varphi(A \times [0, \tau])$, we have index $C$ = index $A$.

The proof of this theorem will make use of the following result.

Proposition 3.15: Suppose $N^n$ is a manifold and $X \subset N^n$ is a compact subset of $N^n$ separating $N^n$, say $N^n \setminus X = U | V$, so that $\bar{U} \cap \bar{V} = X$. Let $A$ denote a compact space, $I = [0,1]$, and $\varphi : A \times I \to N^n$ an imbedding such that $\varphi(A \times \{0\}) \subset U$ and $\varphi(A \times \{1\}) \subset V$. If we set

$$C = \varphi(A \times I) \cap X, \quad g = \mathrm{proj}_1 \circ \varphi^{-1} : \varphi(A \times I) \to A,$$

and $g_0 = g | C$, then

$$g_0^* : H^q(A) \to H^q(C)$$

is injective (one to one) for all $q \geq 0$ (any coefficients).

Proof: There is no loss in identifying $A$ and $\varphi(A \times \{0\})$ and also assuming that $\varphi(a, 0) = a$, $a \in A$. We introduce the notation:

$$B = \varphi(A \times 1)$$

$$A' = \bar{U} \cap \varphi(A \times I)$$

$$B' = \bar{V} \cap \varphi(A \times I)$$

and notice that

$$A' \cup B' = \varphi(A \times I), \qquad A' \cap B' = C .$$

Furthermore, the inclusion maps

$$A \xrightarrow{\alpha} A' \cup B', \qquad B \xrightarrow{\beta} A' \cup B'$$

are homotopy equivalences and $g_0$ serves as a homotopy inverse for $\alpha$. We also introduce the inclusion maps,

$$\begin{array}{ll} i_1 : A' \to A' \cup B', & i_2 : B' \to A' \cup B' \\ j_1 : A' \cap B' \to A, & j_2 : A' \cap B' \to B' \\ k_1 : A \to A', & k_2 : B \to B' . \end{array}$$

Then, $i_1 \cdot k_1 = \alpha$ and $i_2 \cdot k_2 = \beta$ implies the induced maps $i_1^*$ and $i_2^*$ on cohomology are both injections. Consider now the Mayer-Vietoris sequence for $A' \cup B'$,

$$\longrightarrow H^q(A' \cup B') \xrightarrow{\zeta} H^q(A') \oplus H^q(B') \xrightarrow{\eta} H^q(A' \cap B') \longrightarrow$$

where $\zeta = (i_1^*, -i_2^*)$ and $\eta = j_1^* + j_2^*$. This forces $j_1^* : H^q(A') \to H^q(A' \cap B')$ to be an injection as follows. Suppose $j_1^*(a') = 0$. Then, for some $y \in H^q(A' \cup B')$ we have

$$\zeta(y) = (a', 0) = (i_1^*(y), -i_2^*(y))$$

and hence $i_2^*(y) = 0$. This forces $y = 0$ and hence $a' = 0$. Now, consider the retraction $g_1 = gi_1$ of $A'$ to $A$. Since $g_1 k_1 = id_{A'}$, $g_1^*$ is an injection and hence the diagram

$$\begin{tikzcd} H^q(A') \arrow[rr, "j_1^*"] & & H^q(A' \cap B') \\ & H^q(A) \arrow[ul, "g_1^*"] \arrow[ur, "g_0^*"'] & \end{tikzcd}$$

shows that $g_0^*$ is an injection.

Proof of Theorem 3.14: Let $N^n$ denote $\mathbb{R}^n \setminus \{0\}$ with antipodal points identified and apply Proposition 3.15 in $N^n$ with $X = M^{n-1}$ as follows. Set

$$g = \mathrm{proj}_1 \cdot \varphi^{-1} : \varphi(A \times I) \to A$$
$$C = M^{n-1} \cap \varphi(A \times I) .$$

Let $\tilde{A}$, $\tilde{C}$, $\tilde{g}$ denote the corresponding objects in $N^n$ and by Proposition 3.15

$$\tilde{g}_0^* : H^q(\tilde{C}) \to H^q(\tilde{A})$$

is injective. Take a classifying map $(f, \tilde{f})$ for $A$ and we obtain a diagram

$$\begin{array}{ccccc} C & \xrightarrow{g_0} & A & \xrightarrow{f} & S^N \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{C} & \xrightarrow{\tilde{g}_0} & \tilde{A} & \xrightarrow{\tilde{f}} & P^N \end{array}$$

and $\tilde{f}^*(u^k) \neq 0$ if, and only if, $\tilde{g}_0^* \tilde{f}^*(u^k) \neq 0$ and the theorem follows.

Remark 3.16: Proposition 3.15 may also be employed to give an alternative proof of Proposition 3.13.

We indicated at the beginning of this section that this notion of index is equivalent in a restricted category to that introduced by Yang in [11]. We develop this further now.

Let $\mathcal{K}$ denote the category whose objects are pairs $(X, T)$ with $X$ a compact Hansdorff space and $T$ a fixed point free involution on $X$, and whose morphisms are equivariant maps. The following definition is an equivalent formulation of Yang's index (see [11], § 3.6).

<u>Definition 3.17</u>: Given $(X, T) \in \mathcal{K}$, the Yang index of $(X, T)$, denoted by Yang index $X$, is the largest integer $n$ such that for any equivariant map $f : X \to Y$, with $(Y, S) \in \mathcal{K}$ arbitrary,

$$f_* : H_n(\tilde{X}) \to H_n(\tilde{Y})$$

is non-trivial, using Čech homology with $\mathbb{Z}_2$-coefficients, where $\tilde{X}$ and $\tilde{Y}$ are the orbit spaces $X/T$, $Y/S$, respectively.

<u>Proposition 3.18</u>: For $(X, T) \in \mathcal{C}$

$$\text{Yang index } X = \text{index } X .$$

<u>Proof</u>: The proof will make use of duality in Čech theory ([13], [15]) which takes the following form. On the category of compact spaces $X$, there are natural transformations $\varphi$ and $\psi$

$$H^q(X) \xrightarrow{\varphi} [H^q(X)]^* \xrightarrow{\psi} H_q(X)$$

which are isomorphisms for each $X$, where $[H^q(X)]^*$ is the dual over $\mathbb{Z}_2$ of $H^q(X)$. We also make use of the fact that if $(Y, T) \in \mathcal{K}$, there is a finite complex $K$ which admits a free $\mathbb{Z}_2$-action and an equivariant map $h : Y \to K$. $K$ is, in fact, the nerve of an appropriate finite cover of $Y$ and $h$ a barycentric mapping (see [11]).

Now, suppose $(X, T) \in \mathcal{C}$ and $(Y, S) \in \mathcal{K}$ and let $f : X \to Y$ be an equivariant map. Then we have a diagram

$$\begin{array}{ccccc}
H^q(\tilde{X}) & \xrightarrow{\varphi} & [H^q(\tilde{X})]^* & \xrightarrow{\psi} & H_q(\tilde{X}) \\
\uparrow f^* & & \downarrow (f^*)^* & & \downarrow f_* \\
H^q(\tilde{Y}) & \xrightarrow{\varphi} & [H^q(\tilde{Y})]^* & \xrightarrow{\psi} & H_q(\tilde{Y})
\end{array} .$$

If $f_* \neq 0$ for every $Y$, then this is so far $Y = S^N$ and $f^*(u^q) \neq 0$ for $\tilde{Y} = P^N$. Thus, index $X \geq$ Yang index $X$. On the other hand, to show index $X \leq$ Yang index $X$, suppose $Y$ is chosen so that $f_* = 0$. First choose $K$ as above and an equivariant map $g : Y \to K$ and then an equivariant map $h : K \to S^N$ for $N$ sufficiently large. Now, $(h\ g\ f)_* = h_* g_* f_* = 0$ and hence $(h\ g\ f)^* = 0$, where

$$(h\ g\ f)^* : H^q(P^N) \to H^q(\tilde{X}) .$$

This shows, index $X \leq$ Yang index $X$ and the proof is complete.

Let us recall the notion of genus which may be derived from Yang's notion of B-index (or the notion of coindex of Conner-Floyd). Given $(X, T) \in \mathcal{K}$, B-index $X$ is the minimum $k$ such that $X$ admits an equivariant map $f : X \to S^k$. Then, we have, for $(X, T) \in \mathcal{C}$,

$$\text{Yang index } X = \text{index } X \leq \text{B-index } X .$$

Furthermore, for any symmetric compact subset $X$ in a linear space, we have (directly from definitions)

$$\text{genus } X = \text{B-index } X + 1 .$$

It is, therefore, convenient to increase the index by 1 and define the notion of Index $X$ as follows.

<u>Definition 3.19</u>: For $(X, T) \in \mathcal{C}$, set

$$\text{Index } X = \text{index } X + 1 .$$

Remark 3.20: Clearly then

$$\text{Index } X \le \text{genus } X$$

and we note that in [10] Yang has an example of a symmetric imbedding of a polyhedron $K$ in $\mathbb{R}^4$ such that

$$\text{Yang index } K = 1, \quad \text{B-index } K = 2 .$$

Since Yang index $K$ = index $K$ (by Theorem 3.17) we see that

$$\text{Index } K < \text{genus } K$$

so that the Index we have introduced may be strictly less than genus.

Finally one can translate the above relationships to those between Ljsternik-Schnirelman category and Index using the equivalence between genus and category in the appropriate setting (see [9]).

Lemma 2.8 was stated in terms of "Index". Basically the propositions we proved for "index" remain valid for "Index" with minor arithmetic changes. For example,

(3.5)' $X \neq \phi$ implies Index $X \geq 1$ and Index $(\phi) = 0$.

(3.6)' Index $X \leq \dim X + 1$ .

(3.12)' Index $(A \cup B) \leq$ Index $A$ + Index $B$ .

(3.13)' Index $B = n + 1$, where $B$ is the boundary of a symmetric bounded open neighborhood of $0$ in $\mathbb{R}^{n+1}$, e.g. Index $S^n = n + 1$, $n \geq 0$.

Thus, the material in this section constitutes a proof of Lemma 2.8.

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A bifurcation theorem is proved for odd potential operators. The operator equation (*) $f'(u) \equiv Lu + H(u) = \lambda u$ is treated where $\lambda \in \mathbb{R}$ and $u \in E$, a real Hilbert space. A sharp description is given of the structure of the set of solutions of (*) near a bifurcation point as a function of $\lambda$. A crucial role is played here by a notion of topological index alternative to other indices used in critical point theory and the properties of this index are developed in some detail.

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